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ON RIGGED IMMERSIONS

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ABSTRACT

A self-contained account is given in an efficient formalism of rigged immersions of one manifold-with-connection in another, leading to the analogues of the Gauss, Codazzi and Ricci equations discovered by Schouten. The equations expressing their interdependence are then derived and it is shown that in general one of the two sets of "Codazzi" equations is a consequence of the other set and the Gauss and Ricci equations. The formalism is specialised to the Riemannian case, where it is shown that, for large codimension (specific limits being given), all but n components of the Codazzi equations are determined by the other equations. A local theorem on the existence of rigged immersions is proved.

0. Introduction

The interdependence of the Gauss, Codazzi and Ricci equations for isometric immersions has been studied by Blum [1] and others, with emphasis on cases where the embedding has low codimension. Our aim here is to extend this work to higher codimension (with emphasis on the generic case) and to rigged immersions of manifolds having only a connection. As far as I am aware, the only available treatments of the rigged situation are found in Schouten [4] (pp. 265–9) and the references cited therein, whose notation is rather cumbersome for our purposes. We shall therefore develop in sections 1 and 2 the formalism for rigged immersions in a self-contained presentation, enabling us to give in section 3 a very clear proof of the basic equations of Blum on which the interdependence is based. Specialisation to the isometric pseudo-Riemannian case is made in section 4. In section 5 we illustrate the methods used by proving the local existence of a rigged immersion into flat space for a given manifold with a connection, the immersion having its Hölder class as high as is allowed by the Hölder class of the curvature.

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1. Rigged immersions

To begin with, suppose that an n-manifold M is *embedded* in an $(n+p)$-manifold N, so that M can be regarded as a subset of N. All maps, manifolds, bundles etc. will be C^∞. We write the tangent bundles as $TM \subset TN$, with projections π_M and π_N, fibres T_xM and T_xN, and we set $T_MN := \pi_N^{-1}(M) = \bigcup_{x \in M} T_xN$.

Suppose now that at each point $x \in M$ there is given a subspace E_x of T_xN such that $T_xN = E_x \oplus T_xM$ and such that $E = \bigcup_x E_x$ is a smooth subbundle of T_MN. Slightly modifying Schouten's terminology, we call such a bundle a *rigging* of the embedding $M \subset N$.

If M is merely immersed in N by a map $f : M \to N$ then we must modify the definition to say that a rigging is in this case a vector bundle (E, π_E) over M and a smooth fibre map $\bar{f} : E \to TN$, linear and injective on the fibres, such that $\pi_N \circ \bar{f} = f \circ \pi_E$ and $T_{f(x)}N = \bar{f}(E_x) \oplus f_*(T_xM)$. However, since we are concerned here with the *local* properties of immersions rigged in this way, we may as well suppose f to be an embedding, in the sense that locally we can always regard a small enough neighbourhood of M as a subspace of N.

The structure of a rigged embedding derives from the projections of T_xN, for $x \in M$, into its parts tangential to M and transverse to M in E. These we write as

$$\tau : T_MN \to TM \quad \text{and} \quad \nu : T_MN \to E.$$

(In the case where N is pseudo-Riemannian and the induced metric on M is non-degenerate, there is a natural rigging with E the normal bundle, when τ and ν are the orthogonal projections on the tangent and normal bundles.)

If N is furnished with a torsion-free linear connection $\bar{\nabla}$, then we can define an induced connection on M in the same way as in the pseudo-Riemannian case, whether or not the connection is metric or E is normal, by defining

$$\nabla_X Y = \tau(\bar{\nabla}_X Y) \tag{1.1}$$

for all $X \in TM$ and vector fields Y defined on M in a neighbourhood of $\pi_M(X)$ (noting that $\bar{\nabla}_X Y$ can be defined by extending Y to a neighbourhood in N and is independent of the extension chosen).

To describe the local geometry of the immersion, choose sections $\underset{1}{E}, \cdots, \underset{p}{E}$ of E that form a basis in each fibre; define also dual 1-forms $\overset{1}{\omega}, \cdots, \overset{p}{\omega} \in \mathfrak{T}^*_MN$ by requiring $\overset{\alpha}{\omega}(\underset{\beta}{E}) = \delta^\alpha_\beta$, $\overset{\alpha}{\omega}(X) = 0$ for $X \in TM$. Then $\nu(X) = \overset{\alpha}{\omega}(X)\underset{\alpha}{E}$ and $\tau(X) = X - \nu(X)$, so that (1.1) becomes

$$(1.2)\qquad \nabla_X Y = \bar{\nabla}_X Y - \overset{\alpha}{\omega}(\bar{\nabla}_X Y)\underset{\alpha}{E}$$

($X \in TM$, Y a vector field on M). Since $\overset{\alpha}{\omega}(Y) = 0$, we have

$$(1.3)\qquad \overset{\alpha}{\omega}(\bar{\nabla}_X Y) = -(\bar{\nabla}_X \overset{\alpha}{\omega})(Y) =: B^\alpha(X, Y),$$

say, defining tensors B^α analogous to the second fundamental form of Riemannian geometry. Thus (1.2) becomes

$$(1.4)\qquad \nabla_X Y = \bar{\nabla}_X Y - B^\alpha(X, Y)\underset{\alpha}{E}.$$

An analogous formula for covectors (1-forms) can be written down as follows. If $\phi \in \mathfrak{X}^* M$ is a 1-form on M, we write $\tilde{\phi}$ for its extension to $T_M N$ defined by setting $\tilde{\phi}(P) = 0$ for $P \in E$. Then, for $X \in TM$,

$$(\bar{\nabla}_X \tilde{\phi})(Y) = X(\tilde{\phi}(Y)) - \tilde{\phi}(\bar{\nabla}_X Y) = X(\phi(Y)) - \phi(\nabla_X Y) = (\nabla_X \phi)(Y)$$

(from (1.4)) and

$$(1.5)\qquad (\bar{\nabla}_X \tilde{\phi})(\underset{\alpha}{E}) = -\tilde{\phi}(\bar{\nabla}_X \underset{\alpha}{E}) = \phi(\Omega_\alpha(X))$$

say, where $\Omega_\alpha(X) := -\tau(\bar{\nabla}_X \underset{\alpha}{E})$ is the shape operator of ordinary differential geometry. Combining these,

$$(1.6)\qquad \widetilde{\nabla_X \phi} = \bar{\nabla}_X \tilde{\phi} - \tilde{\phi}(\Omega_\alpha(X))\overset{\alpha}{\omega}.$$

Finally, projection onto E defines a connection D in E by

$$D_X P = \nu(\bar{\nabla}_X P) \qquad (X \in TM,\ P : M \to E \text{ a section}).$$

The components of D in the basis $(\underset{\alpha}{E})_{\alpha=1}^{p}$ are given by

$$(1.7)\qquad \sigma_\beta^\alpha(X) := \overset{\alpha}{\omega}(\bar{\nabla}_X \underset{\beta}{E}).$$

Combining (1.7) and (1.5) gives

$$(1.8)\qquad \bar{\nabla}_X \underset{\alpha}{E} = \sigma_\alpha^\beta(X)\underset{\beta}{E} - \Omega_\alpha(X).$$

Those properties which can be defined without a metric are, on the whole, preserved. In particular

$$(1.9)\qquad B^\alpha(X, Y) = B^\alpha(Y, X)$$

follows from the Frobenius relations for the integrability of the distribution defined by the $\overset{\alpha}{\omega}$, which implies that ∇ is torsion-free.

2. Fundamental equations

THEOREM 1 (Schouten). *The following equations hold (where R, $\bar{R}$ are the Riemann tensors of ∇, $\bar{\nabla}$ respectively).*

$$\tau(\bar{R}(X,Y)Z) = R(X,Y)Z - B^{\alpha}(Y,Z)\Omega_{\alpha}(X) + B^{\alpha}(X,Z)\Omega_{\alpha}(Y), \tag{2.1}$$

$$\begin{aligned}\overset{\alpha}{\omega}(\bar{R}(X,Y)Z) &= (\nabla_X B^{\alpha})(Y,Z) - (\nabla_Y B^{\alpha})(X,Z)\\ &\quad + \sigma^{\alpha}_{\beta}(X)B^{\beta}(Y,Z) - \sigma^{\alpha}_{\beta}(Y)B^{\beta}(X,Z),\end{aligned} \tag{2.2}$$

$$\tau(\bar{R}(Y,X)\underset{\beta}{E}) = (\nabla_X\Omega_{\beta})(Y) - (\nabla_Y\Omega_{\beta})(X) + \sigma^{\alpha}_{\beta}(Y)\Omega_{\alpha}(X) - \sigma^{\alpha}_{\beta}(X)\Omega_{\alpha}(Y), \tag{2.3}$$

$$\begin{aligned}\overset{\alpha}{\omega}(\bar{R}(X,Y)\underset{\beta}{E}) &= (\nabla_X\sigma^{\alpha}_{\beta})(Y) - (\nabla_Y\sigma^{\alpha}_{\beta})(X) + B^{\alpha}(Y,\Omega_{\beta}(X))\\ &\quad - B^{\alpha}(X,\Omega_{\beta}(Y)) + \sigma^{\alpha}_{\gamma}(X)\sigma^{\gamma}_{\beta}(Y) - \sigma^{\alpha}_{\gamma}(Y)\sigma^{\gamma}_{\beta}(X).\end{aligned} \tag{2.4}$$

PROOF (1). These follow from the equations of §1 exactly as in the Riemannian case: (2.1) is Gauss' equation, (2.2) and (2.3) correspond to Codazzi's equation and (2.4) is Ricci's equation. Explicitly, (2.1) and (2.2) come from rewriting

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z$$

(for $[X, Y] = 0$) using (1.4), (1.8) and taking tangential and transverse parts; (2.3) follows similarly from

$$-\tilde{\phi}(\bar{R}(X,Y)(\,.\,)) = \bar{\nabla}_X\bar{\nabla}_Y\tilde{\phi} - \bar{\nabla}_Y\bar{\nabla}_X\tilde{\phi}$$

while (2.4) follows on replacing $\tilde{\phi}$ by $\overset{\alpha}{\omega}$. □

The equations can be expressed most succinctly by introducing fields $(\underset{i}{E})_{i=1}^{n}$ locally on M forming a basis in each T_xM, with dual 1-forms $\overset{i}{\omega}$, allowing us to define matrices of 1-forms $\boldsymbol{B}$, $\boldsymbol{\Omega}$, $\boldsymbol{\sigma}$ with components

$$[\boldsymbol{B}]^{\alpha}_{i} = B^{\alpha}(\underset{i}{E},.), \quad [\boldsymbol{\Omega}]^{i}_{\alpha} = \overset{i}{\omega}(\Omega_{\alpha}(\,.\,)), \quad [\boldsymbol{\sigma}]^{\alpha}_{\beta} = \sigma^{\alpha}_{\beta}.$$

The curvature 2-forms are defined as usual by

$$[\boldsymbol{\Theta}]^{i}_{j} = \tfrac{1}{2}\overset{i}{\omega}(R(\,.\,,.\,)\underset{j}{E}) \qquad (i,j = 1,\cdots,n),$$

$$[\bar{\boldsymbol{\Theta}}]^{a}_{b} = \tfrac{1}{2}\overset{a}{\omega}(\bar{R}(\,.\,,.\,)\underset{b}{E}) \qquad (a,b = 1,\cdots,n+p)$$

(this last being considered as restricted to TM).

Then if Z is a vector field on N over M ($Z : M \to T_MN$) we can decompose Z into $\tau(Z) + \nu(Z)$, the two parts having components Z^{Mi} and $Z^{E\alpha}$ with respect to $(\underset{i}{E})$ and $(\underset{\alpha}{E})$. So, for $X \in TM$,

$$\begin{aligned}\bar{\nabla}_X Z &= \bar{\nabla}_X (Z^{Mi} E + Z^{E\alpha} \underset{\alpha}{E}) \\ &= \nabla_X (Z^{Mi} E) + B^{\alpha}(X, Z^{Mi} \underset{i}{E}) \underset{\alpha}{E} \\ &\quad + (\bar{\nabla}_X Z^{E\alpha}) \underset{\alpha}{E} + \sigma^{\beta}_{\alpha}(X) Z^{E\alpha} \underset{\beta}{E} - \Omega_{\alpha}(X) Z^{E\alpha}\end{aligned} \tag{2.5}$$

from (1.8), (1.4). This enables us to give an alternate proof of (2.1)–(2.4).

PROOF (2). Introduce the matrix notation

$$\boldsymbol{Z} = \begin{pmatrix} \boldsymbol{Z}^M \\ \boldsymbol{Z}^E \end{pmatrix}, \quad \boldsymbol{Z}^M = (\boldsymbol{Z}^{Mi})_{i=1}^n, \quad \boldsymbol{Z}^E = (\boldsymbol{Z}^{E\alpha})_{\alpha=1}^p$$

for the components of vectors in $T_M N$, and write $\boldsymbol{d}$ for *covariant* exterior differentiation of tensor valued p-forms on M with respect to the connection ∇ (treating the Greek indices as labels), with $\bar{\boldsymbol{d}}$ for covariant exterior differentiation with respect to $\bar{\nabla}$, using all the indices as tensor indices. Then (2.5) becomes

$$\bar{\boldsymbol{d}}\boldsymbol{Z} = \boldsymbol{d}\boldsymbol{Z} + \begin{pmatrix} 0 & -\boldsymbol{\Omega} \\ \boldsymbol{B} & \boldsymbol{\sigma} \end{pmatrix} \wedge \boldsymbol{Z}$$

or

$$\bar{\boldsymbol{d}}\boldsymbol{Z} = \boldsymbol{d}\boldsymbol{z} + \boldsymbol{F} \wedge \boldsymbol{Z} \tag{2.6}$$

where

$$\boldsymbol{F} = \begin{pmatrix} 0 & -\boldsymbol{\Omega} \\ \boldsymbol{B} & \boldsymbol{\sigma} \end{pmatrix}. \tag{2.7}$$

A second exterior differentiation gives

$$\begin{aligned}\bar{\boldsymbol{\Theta}} \wedge \boldsymbol{Z} = \bar{\boldsymbol{d}}^2 \boldsymbol{Z} &= \boldsymbol{d}^2 \boldsymbol{Z} + \boldsymbol{dF} \wedge Z - \boldsymbol{F} \wedge \boldsymbol{dZ} + \boldsymbol{F} \wedge \boldsymbol{dZ} + \boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{Z} \\ &= \boldsymbol{\Theta}^* \wedge \boldsymbol{Z} + \boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{Z} + \boldsymbol{dF} \wedge \boldsymbol{Z}\end{aligned}$$

i.e.

$$\bar{\boldsymbol{\Theta}} = \boldsymbol{\Theta}^* + \boldsymbol{F} \wedge \boldsymbol{F} + \boldsymbol{dF} \tag{2.8}$$

where

$$\boldsymbol{\Theta}^* = \begin{pmatrix} \boldsymbol{\Theta} & 0 \\ 0 & 0 \end{pmatrix}.$$

Substituting from (2.7) thus gives

$$\bar{\boldsymbol{\Theta}} = \begin{pmatrix} \boldsymbol{\Theta} - \boldsymbol{\Omega} \wedge \boldsymbol{B} & -\boldsymbol{d\Omega} - \boldsymbol{\Omega} \wedge \boldsymbol{\sigma} \\ \boldsymbol{dB} + \boldsymbol{\sigma} \wedge \boldsymbol{B} & \boldsymbol{d\sigma} - \boldsymbol{B} \wedge \boldsymbol{\Omega} + \boldsymbol{\sigma} \wedge \boldsymbol{\sigma} \end{pmatrix}. \tag{2.9}$$

The four blocks of this equation are precisely the four equations (2.1)–(2.4). □

REMARK. Equations (2.1)–(2.4) with $\bar{R} = 0$ are in fact sufficient conditions for the existence of a rigged embedding in $\mathbf{R}^n = N$. The proof of this is almost a word-for-word repetition of the Riemannian proof (see, for example, Tenenblatt [5] for an account that leaves least to the imagination). In fact, the proof is slightly simpler in the present case, since the basis fields are not required to remain orthonormal.

3. Interdependence

The curvature form $\bar{\Theta}$ of $T_M N$ satisfies Bianchi's identity

$$\bar{d}\bar{\Theta} = 0. \tag{3.1}$$

If we note that $\bar{\Theta}$ has components of type $\bar{\Theta}^a_b$, so that its covariant exterior derivative with respect to any connection-forms ω^b_c has components

$$d\bar{\Theta}^a_b + \omega^a_c \wedge \bar{\Theta}^c_b - \bar{\Theta}^a_c \wedge \omega^c_b,$$

then we see that (2.6) extends to forms of the type of $\bar{\Theta}$ to give, for (3.1),

$$0 = \bar{d}\bar{\Theta} = d\bar{\Theta} + F \wedge \bar{\Theta} - \bar{\Theta} \wedge F. \tag{3.2}$$

Generalising from the case where F and $\bar{\Theta}$ arise from a rigged embedding, consider a general matrix of 1-forms

$$f = \begin{pmatrix} 0 & f^i_\beta \\ f^\alpha_i & f^\alpha_\beta \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ b & s \end{pmatrix}$$

and a similar matrix of 2-forms θ satisfying (analogously to (3.2))

$$0 = d\theta + f \wedge \theta - \theta \wedge f. \tag{3.2'}$$

Associated with f and θ is the tensor

$$X := \theta - \Theta^* - f \wedge f - df = \begin{pmatrix} G & C_1 \\ C_2 & K \end{pmatrix}. \tag{3.3}$$

The components G, C_1 and C_2, K are the Gauss, Codazzi and Kühne tensors. Equation (2.8) shows that *the vanishing of X is the necessary condition for (θ, f) to arise from a rigged embedding.*

Exteriorly differentiating (3.3) and using the Bianchi identity $d\Theta = 0$ (and hence $d\Theta^* = 0$) gives

$$\begin{aligned} dX &= d\Theta - df \wedge f + f \wedge df - \Theta^* \wedge f + f \wedge \Theta^* \\ &= -f \wedge \Theta + \Theta \wedge f - df \wedge f + f \wedge df - \Theta^* \wedge f + f \wedge \Theta^* \end{aligned}$$

(from (3.2′)), i.e.,

$$dX = X \wedge f - f \wedge X. \tag{3.4}$$

Thus the components of the embedding equation $X = 0$ (equivalent to (2.8)) are not all independent, since X is constrained by (3.4). It was pointed out by Blum [1] in the pseudo-Riemannian case that (3.4) does not involve θ (except through X). Writing out the components in the block-decomposition of (3.4), we have

$$dG = C_1 \wedge b + \omega \wedge C_2, \tag{3.5}$$

$$dC_1 = C_1 \wedge \varsigma - G \wedge \omega + \omega \wedge K, \tag{3.6}$$

$$dC_2 = K \wedge b - b \wedge G - \varsigma \wedge C_2, \tag{3.7}$$

$$dK = K \wedge \varsigma - C_2 \wedge \omega - b \wedge C_1 - \varsigma \wedge K. \tag{3.8}$$

Suppose now that (θ, f) are such that the Gauss and Ricci equations are satisfied, i.e.

$$G = 0; \qquad K = 0. \tag{3.9}$$

Then (subject still to θ, the putative curvature form, satisfying (3.2′)) equations (3.5) and (3.8) give the algebraic equations

$$C_1 \wedge b + \omega \wedge C_2 = 0, \tag{3.10}$$

$$b \wedge C_1 + C_2 \wedge \omega = 0. \tag{3.11}$$

If these implied that C_1 and C_2 were zero, then the Codazzi equations would be superfluous. However, this is not the case, since (3.10) and (3.11) are clearly satisfied by

$$C_1 = \omega \wedge \phi, \qquad C_2 = b \wedge \phi,$$

where ϕ is any scalar-valued 1-form. (This solution is not possible in the pseudo-Riemannian case because of the relation that exists then between C_1 and C_2.)

It is, however, easy to show that only the Codazzi equation $C_1 = 0$ and the Ricci equation need be satisfied when ω is suitably non-degenerate. To define this, write the components of ω as $\omega^i_{\alpha j} dx^j$, and call ω *maximal* if the map

$$\mathbf{R}^{n^2} \ni (t^j_i)^n_{i,j=1} \mapsto (\omega^i_{\alpha j} t^j_i)^p_{\alpha=1} \in \mathbf{R}^p$$

has rank 2. This will clearly be the case for generic ω and $p \geqq n^2$. While this is possible for a general rigged immersion, it is not possible for an isometric embedding, for which $\omega^i_{\alpha j} = g^{ik}\omega_{\alpha kj}$ with $\omega_{\alpha[kj]} = 0$. We now have

THEOREM 2. *Suppose ω, b, ς and θ are matrices of 1-forms, with entries $\omega^i{}_\alpha$, b^α_i, ς^α_β and θ^a_b ($i = 1, \cdots, n$; $\alpha, \beta = 1, \cdots, p$; $a = 1, \cdots, n+p$), being regarded as tensors of the type indicated by the indices. Let θ satisfy* (3.2′):

$$0 = d\theta + f \wedge \theta - \theta \wedge f,$$

where

$$f := \begin{pmatrix} 0 & -\omega \\ b & \varsigma \end{pmatrix},$$

and suppose that the Ricci equation and one Codazzi equation are satisfied:

$$K = 0; \qquad C_1 = 0,$$

where

$$\begin{pmatrix} G & C_1 \\ C_2 & K \end{pmatrix} := \theta - \Theta^* - f \wedge f - df.$$

Then if ω is maximal, $C_2 = 0 = G$ and all the embedding equations of Theorem 1 are satisfied.

PROOF. (3.6) becomes, in this case,

$$G \wedge \omega = 0$$

or, in components,

$$G^i_{j[kl}\omega^i{}_{\beta m]} = 0 \tag{3.12}$$

(square brackets denoting antisymmetrization over indices of the same type).

The maximality of ω implies the existence of a tensor $\nu^{\beta l}_i$ such that $\nu^{\beta l}_i \omega^j{}_{\beta k} = \delta^j_i \delta^l_k$. Multiplying (3.12) by $\nu^{\beta b}_a$ gives

$$G^i_{j[kl}\delta^b_{m]} = 0$$

and so $G = 0$.

Similarly (3.8) gives

$$C_2 \wedge \omega = 0$$

which implies that $C_2 = 0$, in the same way.

4. The pseudo-Riemannian case

Suppose now that M, N are non-degenerate pseudo-Riemannian manifolds, with M isometrically embedded in N and E the normal bundle. In this case $\tilde{\omega} = \bar{g}(\underset{\beta}{E})\eta^{\alpha\beta}$ ($\bar{g}$ the metric on N), where $\eta_{\alpha\beta} = \bar{g}(\underset{\alpha}{E}, \underset{\beta}{E}) = \pm 1, 0$ for pseudo-orthonormal $\underset{\alpha}{E}$, and $\eta^{\alpha\gamma}\eta_{\gamma\beta} = \delta^\alpha_\beta$. Then Ω and B are related by

(4.1) $$B_i^\alpha = g_{ij}\eta^{\alpha\beta}\Omega_\beta^j$$

(g the metric on M) or

$$\boldsymbol{B} = \boldsymbol{\eta}\boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{g}$$

and similarly $\boldsymbol{C}_2 = -\boldsymbol{\eta}\boldsymbol{C}_1^{\mathrm{T}}\boldsymbol{g}$.

Equations (3.10) and (3.11) then both become equations restricting C_1. It turns out that for large codimension p the most interesting is (3.11) which becomes

$$\boldsymbol{\eta}\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{g} \wedge \boldsymbol{C}_1 - \boldsymbol{\eta}\boldsymbol{C}_1^{\mathrm{T}}\boldsymbol{g} \wedge \boldsymbol{\omega} = 0$$

or, in components, and setting $\boldsymbol{C} \equiv \boldsymbol{C}_1$

(4.2) $$g_{lm}\omega^l_{\alpha[i}C^m_{\beta jk]} - C^m_{\alpha[ij}\omega^l_{\beta k]}g_{lm} = 0.$$

We shall write this as

(4.3) $$C_P A^P_X \equiv C^m_{\beta\langle pq\rangle}A^{\beta\langle pq\rangle}_m\Big|_{\langle\alpha\gamma\rangle\langle ijk\rangle} = 0$$

where

(4.4) $$A^{\beta\langle pq\rangle}_m\Big|_{\langle\alpha\gamma\rangle\langle ijk\rangle} = \delta^\beta_\gamma\,\omega_{m\alpha[i}\delta^p_j\delta^q_{k]} - \delta^\beta_\alpha\,\omega_{m\gamma[i}\delta^p_j\delta^q_{k]}.$$

Here pointed brackets denote the ordering of indices (i.e. $\langle pq\rangle$ means pq or qp according as $p < q$ or $q < p$) and any implied summation or universal quantification is over the range $p < q$, $i < j < k$ etc.; square brackets denote antisymmetrization, and we have lowered indices with g. Capital indices $P, Q, \cdots$ will be used to abbreviate collections of indices of the form $^{\beta\langle pq\rangle}_m$, while $X, Y, \cdots$ will be used for collections of the form $\langle\alpha\gamma\rangle\,\langle ijk\rangle$.

The matrix A with components A^X_P has $\frac{1}{2}n^2p(n-1)$ rows and $\frac{1}{12}np(n-1)\cdot(n-2)(p-1)$ columns. If its rank were equal to the number of rows ("maximal row-rank") then (4.3) would imply that $C_P = 0$, i.e. the Codazzi equations $C_P = 0$ would be implied by the Gauss and Ricci equations (3.9). However, the row-rank of A cannot be maximal (a point apparently overlooked in [1]) since its kernel certainly contains the pn-dimensional subspace spanned by the set $\{C : C^m_{\beta\langle pq\rangle} = \delta^m_{[p}e_{q]}H_\beta,\ \boldsymbol{e} \in \mathbf{R}^n,\ \boldsymbol{H} \in \mathbf{R}^p\}$. There could be further "hidden" relations restricting the rank of A. We shall show, however, that for a large enough number of columns this is the only restriction. Explicitly:

Proposition. *If $[(p-1)/2] \geqq 3n/2$ $(n > 4)$, or if $p \geqq 11$ and $n = 4$, then for $(\omega_{m\alpha i})$ in an open dense set of all such ω with $\omega_{m\alpha i} = \omega_{i\alpha m}$ we have*

(4.5) $$\operatorname{rank}(A) = \tfrac{1}{2}np(n+1)(n-2)$$

(where $[x]$ denotes the largest integer not greater than x).

REMARKS. The condition $[(p-1)/2] \geqq 3n/2$ is satisfied in the general case of a local isometric embedding (for which $p = \frac{1}{2}n(n-1)$) when $n \geqq 8$. For $n = 4$ the condition $p \geqq 1$ is certainly necessary (otherwise there would be insufficient columns). For $n = 5, 6, 7$ computer calculations indicate that it is sufficient to have $[(p-1)/2] \geqq 5, 5, 4$ respectively; in which case a general local isometric embedding would satisfy the condition when $n \geqq 6$. In any case, the proposition is of interest in cases where a high codimension is required in order to secure a good differentiability [3]; but it is of no help if one is interested in algebraically special cases, for instance.

PROOF. First note that, when a matrix depends algebraically on a number of parameters $k_1, \cdots, k_r$ then its rank has its maximum value on the complement of the union of a finite number (possibly zero) of algebraic varieties of dimension less than r. Thus if there is a point $\mathbf{k}$ in a neighbourhood of which the matrix has rank ρ, then ρ will be the generic rank of the matrix. We apply this principle to a sub-block of A in the following lemma.

LEMMA. *Let A' be the matrix with components*

$$A'^{\langle pq \rangle}_{m}\Big|_{\theta \langle ijk \rangle} = \omega_{m\theta[i}\delta^p_j\delta^q_{k]} \tag{4.6}$$

with $\theta = 1, 2, \cdots, s$ and latin indices ranging from 1 to $n \geqq k$. If $s \geqq 3n/2$ $(n > 4)$, or $s \geqq 5$ for $n = 4$, then for generic ω satisfying $\omega_{m\theta i} = \omega_{i\theta m}$ the rank of A' is $\frac{1}{2}n^2(n-1) - n$.

PROOF OF LEMMA. We take first the case $n > 4$.

Suppose s is the smallest integer satisfying the condition. Since $A'^{\langle pq \rangle}_{m}\big|_{\theta\langle ijk \rangle} C^m_{\langle pq \rangle} = 0$ if C lies in the subspace spanned by $\{C : C^m_{\langle pq \rangle} = \delta^m_{[p}e_{q]}, e \in \mathbf{R}^n\}$, it is clear that rank $(A') \leqq \frac{1}{2}n^2(n-1) - n$. Thus, from what was said at the start of the proof of the proposition, it is sufficient to find a set of ω's for which equality holds.

For clarity in what follows, we shall replace the label θ by a pair $\langle ij \rangle$ $(i \neq j)$ and write this pair first; $\langle ij \rangle$ will range over a set P of cardinality s. We then choose the components of the matrix $\omega_{\langle ij \rangle}$ to be

$$\omega_{\langle ij \rangle kl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{4.7}$$

The set P is taken as follows:
For $n = 2k$, even,

$$P = \{\langle 1,2 \rangle, \langle 2,3 \rangle, \cdots, \langle n-1, n \rangle, \langle n, 1 \rangle \langle 1, k+1 \rangle, \langle 2, k+2 \rangle, \cdots, \langle k, 2k \rangle\}.$$

For $n = 2k+1$, odd,

$$P = \{\langle 1,2\rangle, \cdots, \langle n-1,n\rangle, \langle n,1\rangle, \langle 1,k+1\rangle, \langle k,2k\rangle, \langle k+1,2k+1\rangle\}.$$

This ensures the following two properties:

(4.8) $\langle i, i+1\rangle \in P$ and $\langle n,1\rangle \in P$, for $i = 1, \cdots, n-1$;

(4.9) for any ijk $\exists r$ distinct from i, j, k with $\langle jr\rangle \in P$.

With the choice (4.7), A' takes the form

$$A'^{\langle pq\rangle}_{m}\Big|_{\langle ij\rangle\langle rst\rangle} = (\delta_{im}\delta_{j[r} + \delta_{jm}\delta_{i[r})\delta^{t}_{s}\delta^{q}_{t]}.$$

It is clear from this that, for a non-zero component of A', we require either ($m = i$ and $\{p,q,j\} = \{r,s,t\}$) or ($m = j$ and $\{p,q,i\} = \{r,s,t\}$). Thus the non-zero elements of A' can be labelled by the sets $(\langle st\rangle, m, r)$, with r, s and t distinct, there being a correspondence between such sets and the non-zero elements of A' defined by

$$(\langle st\rangle m, r) \mapsto A'^{\langle st\rangle}_{m}\Big|_{\langle mr\rangle\langle rst\rangle} = \pm 1$$

the sign being given by the product of the signatures of the permutations required to order (st) and (rst).

Two distinct elements are in the same column of A' if they have the same pairs $\langle mr\rangle$ and $\langle rst\rangle$; i.e. if their index sets are (st, m, r) and $(s'r', r, m)$ with $\{m, s', t'\} = \{r, s, t\}$. Since $r \neq m$ for $\langle rm\rangle \in P$, this means ($r = s'$ or $r = r'$) and ($m = s$ or $m = t$). The index sets can thus be written (st, s, r) and (rt, r, s). Sets of the form (st, m, r) with $m \notin \{s,t\}$ thus label elements in columns in which only that element is non-zero, while in the other columns there are exactly two non-zero elements.

Let us reorder the rows and columns of A', starting with those rows labelled by $m\langle st\rangle$ with m, s, t all distinct. By our choice of P (4.9), for each of these we can find an r such that $\langle mr\rangle \in P$, with m, r, s, t distinct, giving an isolated non-zero element in the column labelled by $\langle mr\rangle$ $\langle rst\rangle$. Starting with these columns in the order corresponding to the rows reduces A' to the form

$$A' = \left(\begin{array}{c|c} J & \\ \hline 0 & \end{array}\right)$$

where J is diagonal with entries ± 1, of rank $\frac{1}{2}n(n-1)(n-2)$.

Choose an index n and consider the following elements of A', here referred by their index sets and shown along with their row- and column-labels (Fig. 1). There are no other entries in the submatrix K_n defined by these rows and columns, since there are at most two elements in each column; and hence the rank is $n-2$. Moreover, the rows and columns constructed for different values of n are distinct. Thus we can group the rows and columns for $n=1,2,\cdots,n$ so as to reduce A' to the form

$$A' = \left(\begin{array}{c|c|c|c} J & 0 & 0 & 0 \\ \hline 0 & K_1 & 0 & 0 \\ \hline 0 & 0 & & 0 \\ \hline 0 & 0 & 0 & K_n \end{array}\right)$$

(noting that this now covers all the rows).

It follows from this form that

$$\operatorname{rank} A' \leqq \operatorname{rank} J + \sum_{u=1}^{n} \operatorname{rank} K_u$$

$$= \tfrac{1}{2}n(n+1)(n-2) = \tfrac{1}{2}n^2(n-1) - n.$$

Since this is maximal, the lemma is proved for $n > 4$.

The proof for $n = 4$ consists of a verification that the rank is maximal for the above choice of ω, but with $P = \{\langle 12\rangle, \langle 2,3\rangle, \langle 3,4\rangle, \langle 13\rangle, \langle 2,4\rangle\}$.

Proof of Proposition (continued). Choose a general set of $(\overset{0}{\omega}_{m\alpha i})(\overset{0}{\omega}_{m\alpha i} = \overset{0}{\omega}_{i\alpha m})$ and let $\omega_{m\alpha i} = q_\alpha \overset{0}{\omega}_{m\alpha i}$ (no summation over α) where the q_α are a set of parameters to be chosen later. Suppose $[(p-1)/2] = s$. We then arrange the columns of A in groups, each group having fixed $\langle \alpha\gamma\rangle$ and varying i, j, k (cf. (4.5)). The groups will themselves be arranged in the order indicated along the top of Fig. 2, while the rows will be grouped into rows with the same β (cf. (4.5)), in order of increasing β. Figure 2 then describes the non-zero blocks of A (columns labelled by $\langle \alpha\gamma\rangle = \langle r, 2s+2\rangle$ for $r > s$ in the case when p is even are omitted).

The rank is essentially determined by the parts of the matrix enclosed in boxes in Fig. 2. Each of these is of the form of the matrix A of the lemma and so has rank $\frac{1}{2}n(n+1)(n-2)$. If one imagines performing row operations to reduce A to echelon form, it is apparent that, providing $q_1 \gg q_2 \gg \cdots \gg q_p$, then each of these boxed sections will contribute to the rank of A. In this case $\operatorname{rank}(A) = \frac{1}{2}np(n-1)(n-2)$ as required. □

The analogue of Theorem 2 for the pseudo-Riemannian case can now be written down, as follows.

THEOREM 3. *Suppose* $\boldsymbol{\omega}$, $\boldsymbol{\varsigma}$ *and* $\boldsymbol{\theta}$ *are matrices of* 1*-forms with entries* $\omega^i_{\ \alpha}$, ς^α_β *and* θ^a_b. *Define* $b^\alpha_i = \eta^{\alpha\beta} g_{ij} \omega^j_{\ \beta}$ *where* η *is a diagonal matrix with entries* ± 1 *and* g *is a non-degenerate pseudo-Riemannian metric on* M. *Suppose also that* $(\boldsymbol{\omega}, \boldsymbol{b}, \boldsymbol{\varsigma}, \boldsymbol{\theta})$ *satisfy* (1) *of Theorem* 2; *that (in the notation of Theorem* 2) *the Gauss and Ricci equations are satisfied*:

$$\boldsymbol{G} = 0, \qquad \boldsymbol{K} = 0;$$

that the components $b_{\alpha ij}$ *satisfy* $b_{\alpha[ij]} = 0$; *and that at each point* x *of* M $\boldsymbol{C}$ *lies in a subspace of tensors transverse to the space* V_x *generated by matrices of* 2*-forms with entries*

$$C^m_\beta = H_\beta dx^m \wedge \phi_x$$

$(\boldsymbol{H} \in \mathbf{R}^p$, ϕ *a* 1*-form on* M). *If* $[(p-1)/2] \geqq 3n/2$ $(n > 4)$, *or if* $p \geqq 11$ *and* $n = 4$, *then for generic* $\boldsymbol{\omega}$ *the Codazzi equation* $\boldsymbol{C} = 0$ *is also satisfied.*

PROOF. We need merely remark that V_x has already been identified with the kernel of A by the proposition, and $\boldsymbol{CA} = 0$ by (4.3). It follows that $\boldsymbol{C} = 0$. □

Finally, we remark that equation (3.10) can be used further to restrict the avoided subspace V_x. However, the restriction depends on $\boldsymbol{\omega}$ and (for large β) is not such as to make $\boldsymbol{C} = 0$ an inevitable consequence of (3.10) and (3.11).

5. Applications

The motivation for developing this formalism and establishing the dependency results (Theorems 2 and 3) has been the desire to approach general immersion results through a direct solution of the Gauss–Codazzi–Ricci equations. Previously it has been customary, in the pseudo-Riemannian case, to use these equations mainly for the explicit calculation of embeddings of algebraically special spaces, relying on direct manipulation of the equations for the metric in the general case. It is hoped that the proposed alternative approach will allow one to prove the existence of immersions with an optimum differentiability for a given differentiability of the curvature of the immersed manifold (where, as we have already noted in relation to [3], a larger codimension is required than is needed if differentiability conditions are relaxed). This should in turn shed light on the difficult question of how the differentiability of the connection is determined by that of the curvature, where the author has so far obtained only partial results [2].

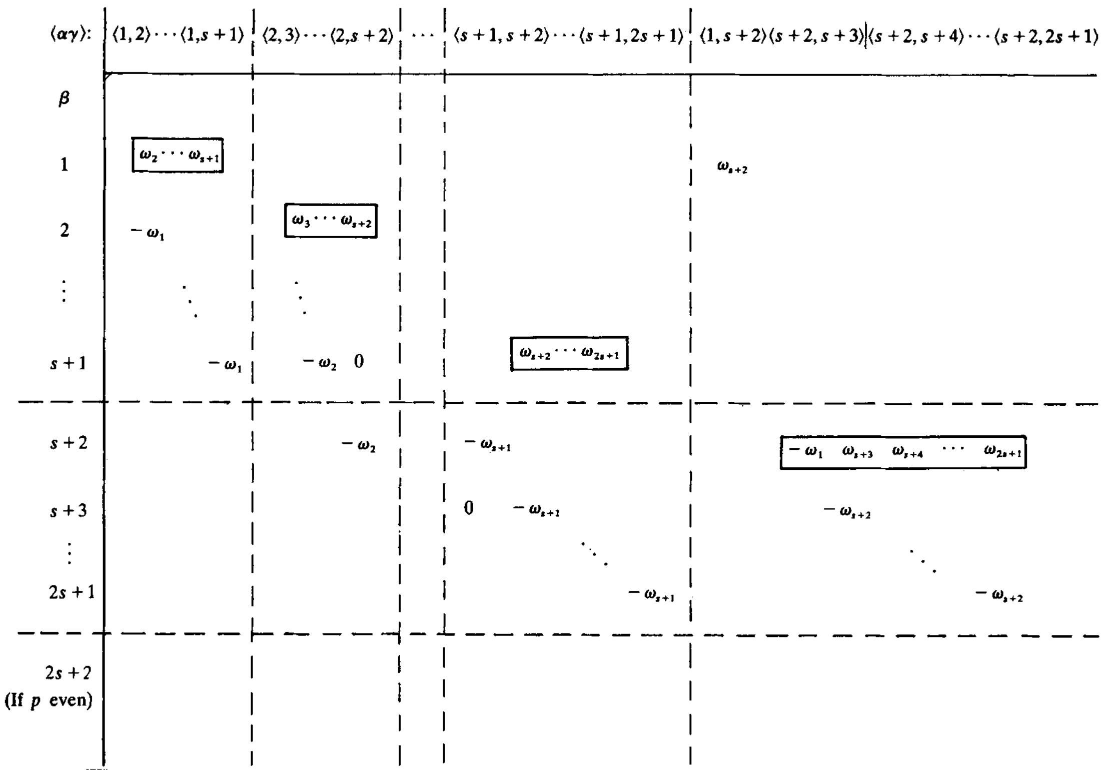


Fig. 2. The matrix A.

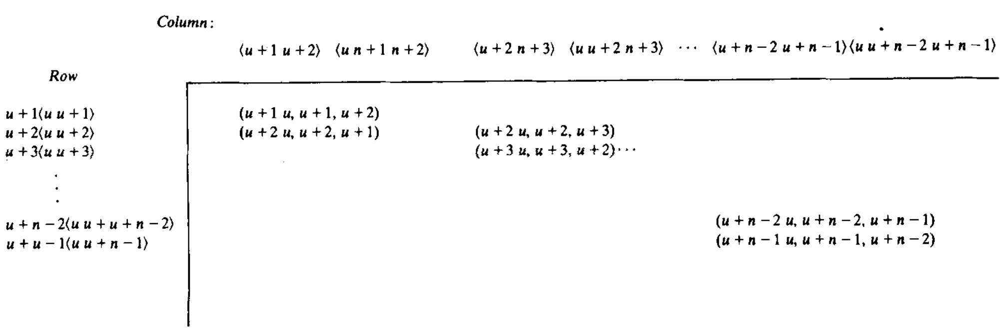
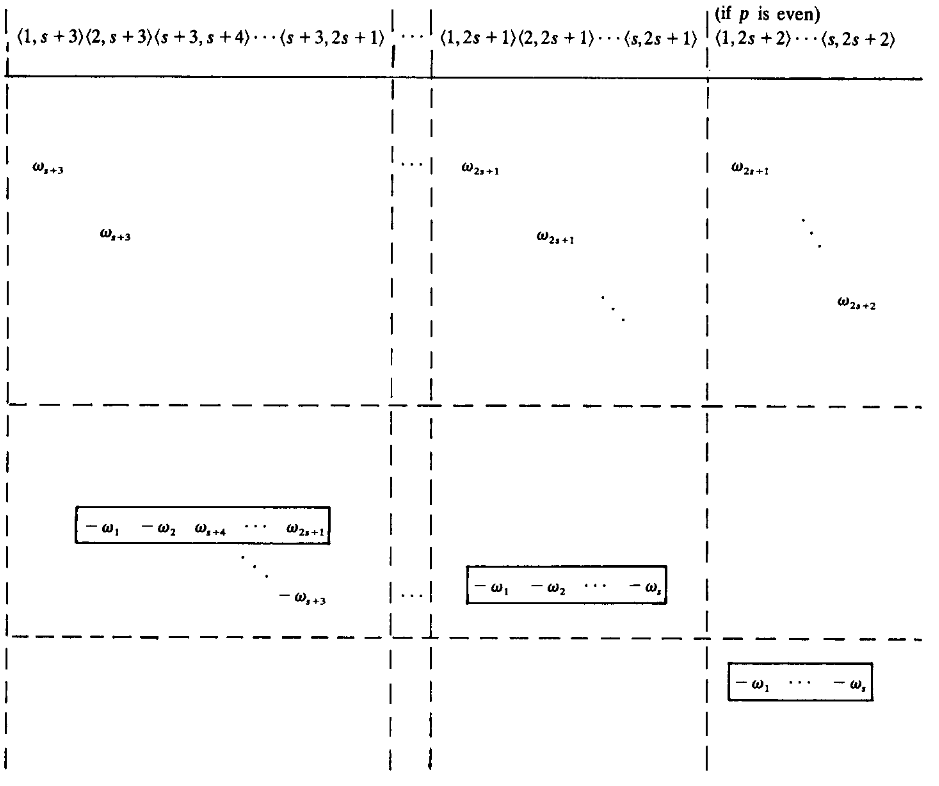


Fig. 1. The matrix K_u. Addition of indices is modulo n.


(if p is even)
$\langle 1, s+3\rangle\langle 2, s+3\rangle\langle s+3, s+4\rangle\cdots\langle s+3, 2s+1\rangle$
$\cdots$
$\langle 1, 2s+1\rangle\langle 2, 2s+1\rangle\cdots\langle s, 2s+1\rangle$
$\langle 1, 2s+2\rangle\cdots\langle s, 2s+2\rangle$
ω_{s+3}
ω_{s+3}
ω_{2s+1}
ω_{2s+1}
ω_{2s+1}
ω_{2s+2}
$-\omega_1 \quad -\omega_2 \quad \omega_{s+4} \quad \cdots \quad \omega_{2s+1}$
$-\omega_{s+3}$
$-\omega_1 \quad -\omega_2 \quad \cdots \quad -\omega_s$
$-\omega_1 \quad \cdots \quad -\omega_s$

As an example, we give a simple local thorem for rigged immersions, using the dependence relation (3.6) that forms the main part of Theorem 2. Though this uses only a small part of the results obtained, it illustrates the potentialities of the approach. As the result is only intended as an illustration, we omit some of the details in the proof.

THEOREM 4. *Let there be given on* $\mathbf{R}^n$ *a differentiable matrix of connection forms* $\boldsymbol{\omega}$ *having a Hölder continuous (exponent* $0 < \alpha < 1$*) matrix of curvature forms. Then the equations* (2.9) *(Gauss–Codazzi–Ricci) with* $\boldsymbol{H} = 0$ *admit a solution for* $p = n^2$ *in a small enough neighbourhood of the origin, with* $\boldsymbol{\Omega}$, $\boldsymbol{B}$ *and* $\boldsymbol{\sigma}$ *Holder continuous (exponent* α*).*

COROLLARY. *Under the assumption of the theorem, there exists a class* $C^{2,\alpha}$ *immersion of a neighbourhood of the origin into* $\mathbf{R}^{n+p}$, *and a* $C^{1,\alpha}$ *rigging, which induce the connection* $\boldsymbol{\omega}$.

PROOF OF THEOREM. We denote the norms corresponding to the classes C^α and $C^{1,\alpha}$, in a domain D with compact closure, by $\|\ \|_D$ and $\|\ \|_D^1$ respectively. When D is a ball of radius R centre the origin the subscript D will be omitted. Our main tool is provided by the following lemmas.

LEMMA 1. *Let* D *be an open domain in* $\mathbf{R}^n$, *star-shaped from all points in a ball of radius* a *centre the origin. Then there exists a linear map* I_a *from* p*-forms on* D *to* $(p-1)$ *forms on* D $(p = 1, 2, \cdots, n)$ *satisfying*

$$dI_a\phi = \phi - I_a d\phi, \tag{5.1}$$

$$\|I_a\phi\|_D^1 \leqq C\|\phi\|_D, \qquad \|I_a\phi\|_D \leqq RC\|\phi\|_D, \tag{5.2}$$

where C *depends on* R, a *and* p *with* $\limsup_{\beta\to 0} C(\beta R, \beta a, p)$ *finite.*

PROOF. See [2], §7.2. The assertion concerning C follows from an obvious scaling argument on noting that if $f: x \mapsto \beta x$ then $f^* I_a\phi = I_{a/\beta} f^*\phi$.

LEMMA 2. *Let* D *be as above and contained in a ball of radius* R *and centre the origin. Let there be given a connection on the bundle* $\mathbf{R}^q \times D \to D$ *with connection* 1*-forms* ω *and curvature* 2*-forms* H *(both matrix-valued); and let*

$$\|\omega^i_j\|_D \leqq \min(1/a, 1/[C(R+a)]) \tag{5.3}$$

where C *is the constant of Lemma* 1. *Then there exists an operator* J_a *taking* $\mathbf{R}^q$*-valued* p*-forms to* $\mathbf{R}^q$*-valued* $(p-1)$*-forms satisfying*

$$dJ_a\phi = \phi - J_a d\phi + J_a\Theta J_a\phi, \tag{5.4}$$

(5.5) $$\| J_a\phi \|_D^1 \leqq C' \| \phi \|_D, \qquad \| J_a\phi \| \leqq RC' \| \phi \|_D,$$

with C' having the properties of C.

PROOF. Define

$$J_a\phi := \sum_{r=0}^{\infty} (-I_a\omega)^r I_a\phi$$

which converges from (5.2) and (5.3). Then clearly

(5.6) $$J_a(1+\omega I_a)\phi = I_a\phi$$

and

(5.7) $$(1+I_a\omega)J_a\phi = I_a\phi.$$

Differentiating (5.7)

$$dJ_a\phi + dI_a\omega J_a\phi = dI_a\phi$$

whence, from (5.1),

$$dJ_a\phi - I_a d\omega J_a\phi + \omega J_a\phi = -I_a d\phi + \phi,$$

i.e.

(5.8) $$dJ_a\phi = \phi - I_a d(\phi - \omega J_a\phi).$$

Covariantly differentiating again

$$\begin{aligned}\Theta J_a\phi = d^2 J_a\phi &= d\phi - dI_a d(\phi - \omega J_a\phi) - \omega I_a d(\phi - \omega J_a\phi)\\ &= d\phi - (1+\omega I_a)d(\phi - \omega J_a\phi)\end{aligned}$$

(using (5.1) again).

Operating with J_a and using (5.6) then gives

$$\begin{aligned}J_a\Theta J_a\phi &= J_a d\phi - I_a d(\phi - \omega J_a\phi)\\ &= J_a d\phi + dJ_a\phi - \phi\end{aligned}$$

(from (5.8)). Rearranging then gives (5.4).

The proof of (5.5) is a direct calculation applying (5.2) to the definition of J_a and performing the summations.

PROOF OF THEOREM (continued). We shall solve (2.8) with $\bar{\Theta} = 0$ by an iterative procedure (i.e. using what is essentially a contraction-mapping argument). From now on D is the R-ball at the origin, $a = R/2$ and subscripts "a"

and "D" will be omitted. At the nth stage of the iteration the variables are

$$\overset{n}{F} = \begin{pmatrix} 0 & -\overset{n}{\mathbf{\Omega}} \\ \overset{n}{\mathbf{B}} & \overset{n}{\boldsymbol{\sigma}} \end{pmatrix}$$

with the initial values

$$\overset{0}{\mathbf{B}} := 0, \qquad \overset{0}{\boldsymbol{\sigma}} := 0,$$

$$\overset{0}{\Omega}{}^{m}_{(j-1)n+ik} := -\delta^{i}_{k}\delta^{m}_{j}K \tag{5.9}$$

($i, j = 1, \cdots, n$; K a constant determined later).

These correspond to the canonical immersion of flat $\mathbf{R}^n$ in $\mathbf{R}^{n+n^2}$ with a non-trivial rigging.

Iteration is determined by

$$\overset{n}{\boldsymbol{F}} := \overset{n-1}{\boldsymbol{F}} + \Delta\overset{n}{\boldsymbol{F}}, \qquad \Delta\overset{n}{\boldsymbol{F}} = -PJ\overset{n-1}{\boldsymbol{X}} \tag{5.10}$$

where

$$P\begin{pmatrix} A & B \\ C & D \end{pmatrix} := \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

and $\boldsymbol{X}$ is defined by (3.3) with $\bar{\boldsymbol{\Theta}} = 0$.

Since P commutes with the action of J, (3.3), (5.10) and (5.4) give, by direct substitution,

$$\begin{aligned} \overset{n}{\boldsymbol{X}} = \overset{n-1}{\boldsymbol{X}} - P\overset{n-1}{\boldsymbol{X}} + (PJ\overset{n-1}{\boldsymbol{F}}) \wedge \overset{n-1}{\boldsymbol{X}} + PJ\boldsymbol{\Theta} * J\overset{n-1}{\boldsymbol{X}} \\ - \overset{n-1}{\boldsymbol{F}} \wedge PJ\overset{n-1}{\boldsymbol{X}} + (PJ\overset{n-1}{\boldsymbol{X}}) \wedge PJ\overset{n-1}{\boldsymbol{X}} \end{aligned} \tag{5.11}$$

so that

$$P\overset{n}{\boldsymbol{X}} = P((PJ\overset{n-1}{\boldsymbol{F}}) \wedge \overset{n-1}{\boldsymbol{X}}) + \text{similar terms.} \tag{5.11$'$}$$

This last equation very nearly establishes the contractive property of the iteration, except that on the right-hand side there appears $\overset{n-1}{\boldsymbol{X}}$ instead of $P\overset{n-1}{\boldsymbol{X}}$. We circumvent this by using the dependency relation (3.7) and the proof of Theorem 2 to relate the Gauss-equation error (the part of $\boldsymbol{X}$ not appearing in $P\boldsymbol{X}$) to the rest of $\boldsymbol{X}$.

Suppose that

$$\|\overset{n}{\mathbf{\Omega}} - \overset{0}{\mathbf{\Omega}}\| < K/2$$

so that $\overset{n}{\Omega}$ is maximal. (This holds for $n = 0$ — cf. (5.9); if we take it to be true for $n < N$ then after the contractive property has been established it will become clear that we can take $N = \infty$ provided K is large enough.) Thus, proceeding as in the proof of Theorem 2 we obtain

$$(G \wedge \boldsymbol{\omega})\boldsymbol{\nu} = (\boldsymbol{C}_1 \wedge \boldsymbol{\sigma} + \boldsymbol{\omega} \wedge \boldsymbol{K} - \boldsymbol{dC}_1)\boldsymbol{\nu}$$

(where $\boldsymbol{\nu}$, the inverse of $\boldsymbol{\omega}$ as defined in Theorem 2, is unique for $p = n^2$) leading to

$$\|\overset{(n)}{\boldsymbol{G}}\| \leqq (K^{-1}\|P\overset{n}{\boldsymbol{X}}\|^1 + \|P\overset{n}{\boldsymbol{X}}\|) \times \text{const.} \tag{5.12}$$

the constant depending on $\overset{n}{\boldsymbol{F}}$.

We next evaluate the first term on the r.h.s. by applying (5.5) to (5.11)′, differentiated, obtaining an inequality of the form

$$\begin{aligned} \|P\overset{(n)}{X}\|^1 &\leqq \text{const.} \times R(\|\overset{n-1}{\boldsymbol{F}}\|\,\|\overset{n-1}{\boldsymbol{X}}\|^1 + \|\overset{n-1}{\boldsymbol{F}}\|^1\|\overset{n-1}{\boldsymbol{X}}\|) \\ &\quad + \text{const.} \times \|\overset{n-1}{\boldsymbol{X}}\|(\|\boldsymbol{F}\| + \|\boldsymbol{H}\| + \|\overset{n-1}{\boldsymbol{X}}\|) \end{aligned} \tag{5.13}$$

(where the constants, derived from C' in (5.5), are bounded as $R \to 0$ with R/a fixed).

Moreover, from (5.10) we have

$$\|\overset{n}{\boldsymbol{F}}\|^1 \leqq (\|\overset{n-1}{\boldsymbol{F}}\|^1 + \|\overset{n-1}{\boldsymbol{X}}\|) \times \text{const.}$$

so that

$$\|\overset{n}{\boldsymbol{F}}\|^1 \leqq \text{const.} \times \left(\sum_{r=1}^{n-1} \|\boldsymbol{X}\| + \|\overset{0}{\boldsymbol{F}}\|^1\right).$$

We shall carry out the iteration with the summation and $\|\overset{n}{\boldsymbol{F}}\|$ bounded; thus $\|\overset{n}{\boldsymbol{F}}\|^1$ will be bounded and (5.13) can be written

$$\|P\overset{n}{\boldsymbol{X}}\|^1 \leqq \text{const.} \times (R\,\|\overset{n-1}{\boldsymbol{X}}\|^1 + \|\overset{n-1}{\boldsymbol{X}}\|).$$

Using this, (5.12) becomes

$$\|\overset{(n)}{\boldsymbol{G}}\| \leqq \text{const.} \times (K^{-1}R\,\|\overset{n-1}{\boldsymbol{X}}\|^1 + K^{-1}\|\overset{n-1}{\boldsymbol{X}}\| + \|P\overset{n}{\boldsymbol{X}}\|). \tag{5.14}$$

A similar argument applied to (5.11) shows that $\|\overset{n}{\boldsymbol{X}}\|^1$ is bounded, in which case (5.14) becomes

$$\|\overset{n}{\boldsymbol{X}}\| \leqq \text{const.} \times (K^{-1}\|\overset{n-1}{\boldsymbol{X}}\| + RK^{-1} + \|P\overset{n}{\boldsymbol{X}}\|). \tag{5.15}$$

Inserting this in (5.11) gives

$$\begin{aligned} \|P\overset{n}{\boldsymbol{X}}\| &\leqq \text{const.} \times \{R(K^{-1}\|\overset{n-1}{\boldsymbol{X}}\| + RK^{-1} + \|P\overset{n}{\boldsymbol{X}}\|) \\ &\quad + R\,\|\boldsymbol{\Theta}\|\,\|P\overset{n-1}{\boldsymbol{X}}\| + P(K^{-1}\|\overset{n-2}{\boldsymbol{X}}\| + RK^{-1} + \|P\overset{n-1}{\boldsymbol{X}}\|)\} \end{aligned}$$

and then inserting this in (5.15) gives, finally,

$$\|\overset{n}{X}\| \leqq \text{const.} \times (K^{-1} + R)(\|\overset{n-1}{X}\| + \|\overset{n-2}{X}\|)$$

for small enough R.

Thus for small enough R and K^{-1} we can prove by induction that $\|\overset{n}{X}\| \to 0$, with the restrictions on $\|\Omega - \overset{0}{\Omega}\|$ and $\Sigma \|\overset{n}{X}\|$ maintained. □

References

1. R. Blum, *Subspaces of Riemannian spaces*, Can. J. Math. **7** (1955), 445.

2. C J S Clarke, *Space-times of low differentiability and singularities*, J. Math. Anal. Appl. (1982), to appear.

3. A. Källén, *Isometric embedding of a smooth compact manifold with a matric of low regularity*, Ark. Mat. **16** (1978), 29–50.

4. J. A. Schouten, *Ricci-calculus*, Springer-Verlag, Berlin–Göttingen–Heidelberg, 1954.

5. K. Tenenblatt, *On isometric immersions of Riemannian manifolds*, Bol. Soc. Brasil. Mat. **2** (1971) 23–36.

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